Summary
The Notion of Mathematical Proof

The monograph is devoted to the philosophical analysis of the methodological and epistemological status of mathematical proofs. Three main groups of problems are discussed:

(1) The relationship between the non-formal mathematical proofs we encounter in mathematical practice and their formalized counterparts (as investigated within the proof theory).

(2) The explanatory role of mathematical proofs and the problem of understanding in mathematics.

(3) The problem of empirical elements in mathematical proofs, in particular the status of computer-assisted proofs and (hypothetical) proofs involving new computational models.

In the monograph, I consider two radically different points of view concerning the nature of mathematical proof. They might be labeled as the *semantic* and the *formalistic* point of view. In the semantic tradition, a mathematical proof is considered to be a sequence of intuitively acceptable propositions. The crucial fact is that competent mathematicians understand the proofs and accept them: they recognize the assumptions and the steps in the proofs as legitimate. From this point of view, proving theorems is a sequence of intellectual acts and doing mathematics is possible because we have a kind of *intelektuelle Anschauung* of the subject matter. From the formalist point of view, on the other hand, a mathematical proof is a formal construct, where purely formal rules of manipulating strings of symbols are given. The intuitive understanding of the proof by the mathematicians is neither a necessary nor a sufficient condition for the correctness of the proof. From this point of view, the essence of proving theorems is therefore performing formal manipulations rather than intellectual acts.

It is a remarkable fact that mathematical proofs we encounter in everyday practice are precise and rigorous. They are not, however, formalized in the sense of proof theory and look rather quite different from their formal counterparts. Practicing mathematics starts not with setting up formal axioms but rather with informal considerations, looking for motivations, identifying crucial notions, formulating important problems to be solved

etc. It cannot be denied that there is always a (inter)subjective element – as certain rules have to be accepted at the pre-theoretic level and the standards of proofs undergo a steady evolution driven by the changes in our intuitive understanding of the notion of legitimate mathematical argumentation. In particular the problem of the relationship between the formal and the informal elements in mathematical proofs (of the so called "Hilbert's bridge" between the informal proof and its formal counterpart) needs to be explained. This leads to an interesting philosophical discussion.

Formalism has a foundational character – in the sense that it has the ambition of formulating criteria of being an acceptable proof. But the proofs we encounter in mathematical practice are not formalized – and these ordinary, non-formal proofs convey mathematical ideas and provide understanding – in general, they produce new mathematical knowledge. It is hard to imagine that mathematicians (with very few exceptions) would like to formulate their proofs as strings of symbols in a formal system. The strong normative claims of formalism are countered by an antifoundationalist reaction, of which a prominent representative is Lakatos. He devotes much attention to the context of discovery and to the dynamics of mathematical notions – the problem, which cannot be adequately addressed within the formalistic framework. The account given by Lakatos is a good starting point for the discussion concerning the explanatory role of mathematical proofs. The problem of explanation in mathematics is an important notion, which occurs often in informal discussions. Surely, there is something more in our notion of mathematical knowledge than just the existence of a formal proof. Usually, we expect proofs not only to prove theorems but also to e x p l a i n why the mathematical facts obtain. The insights offered by the proof are often much more important than the proven theorem itself. The problem of accounting for the explanatory role of proofs becomes more acute, when we consider computer-assisted proofs as well as hypothetical quantum proofs and hypercomputational procedures.

Another group of problems concerns the empirical elements involved in the process of producing mathematical knowledge. Computer-assisted proofs are a case in point (I consider the proof of the four-color theorem here but the considerations apply to other computer-assisted proofs as well). The use of the computer can be considered to be a kind of a physical experiment which needs to be analyzed. This need becomes even more pressing in the case of (hypothetical) quantum-computer-assisted proofs, where additionally certain philosophical problems concerning quantum mechanics are inherited. The question of the explanatory value of such proofs arises, as we have no (even no theoretical) possibility of obtaining insight into the computation (since performing a measurement would destroy the process). The process is thus not even theoretically surveyable, which makes the situation qualitatively quite new.

I also consider a thought experiment, concerning hypercomputation, where we have at our disposal problem-solving devices, which outperform the Turing machine (by providing answers to uncomputable problems like the halting problem). The discussion concerning these models shows that the empirical ingredient (or: resource) depends on the background physical theory – it is different in the case of classical, quantum or relativistic physics. So, if we made use of such empirical devices then the status of the knowledge obtained would become problematic. Could we still call it mathematical knowledge? We would not be able to provide any mathematical argument in favor of the mathematical claims justified by the (hypothetical) hypercomputational procedure – we have to rely on a verdict of an "empirical oracle". The knowledge obtained seems rather to be a kind of quasi-empirical knowledge. I think that the most coherent account could be given within a Quine-style quasi-empiricist standpoint. From this point of view, our mathematical knowledge constitutes a part of our web of belief: mathematical claims are justified not by intuitive access but, ultimately, by the analysis of the relationships between mathematics and science.

Some of the models discussed in the book have a purely theoretical (speculative) character. However, they invite us to rethink the traditional concept of mathematical knowledge, the notion of explanation in mathematics and the problem of empirical elements in proofs. The considerations concerning these models offer new arguments in the realism-antirealism debate.